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Now let $2\sqrt{D+d}$ be the integer next greater than $2\sqrt{D}$ and choose ε so that

$$\varepsilon < \frac{d}{2|a|}, \text{ whence } M < 2\sqrt{D+d}.$$

Hence by inequality (7), since a_{m+1} is an integer,

$$a_{m+1} < 2\sqrt{D}$$

for every $m, m > i$. This is the well known

THEOREM. *In the continued fraction representing either quadratic irrationality $(-b \pm \sqrt{D})/a$, a, b, D integers, $D > 0$, every partial denominator from one of a certain rank on, is less than $2\sqrt{D}$.*

7. The more general theorem of which this is a special case is the following

THEOREM. *In the continued fraction representing the real number ξ , where ξ is the root of an irreducible equation with integral coefficients $f(x) = 0$ of degree n , every partial denominator a_{m+1} , from one of a certain rank on, satisfies the inequality $a_{m+1} < |f'(\xi)| q_m^{n-2}$.*

PROPERTIES OF THE FUNCTION $(1+a)^x$.

By ANTONIO LLANO, Scranton, Pa.

The following demonstrations of some well known theorems are submitted as being simpler and more systematic than those usually given. The binomial $1+a$ is supposed positive, or $a > -1$.

THEOREM I. *If $x > 1$, then $(1+a)^x > 1+ax$.*

Let $x = u/v$, where $u > v$, and put $1+a = z^v$. We have

$$\frac{z^u - 1}{z^v - 1} = \frac{z^{u-1} + z^{u-2} + \dots + 1}{z^{v-1} + z^{v-2} + \dots + 1} = 1 + \frac{z^v + z^{v+1} + \dots + z^{u-1}}{z^{v-1} + z^{v-2} + \dots + 1} \dots (1).$$

If $z > 1$, the fraction in the final member is less than $\frac{(u-v)z^v}{vz^{v-1}}$.

$$\therefore \frac{z^u - 1}{z^v - 1} > 1 + \frac{(u-v)z}{v} > 1 + \frac{u-v}{v}, \text{ namely, } \frac{u}{v} \dots (2).$$

$$\therefore z^u > 1 + (z^v - 1) \frac{u}{v}; \text{ or, } (1+a)^{u/v} > 1 + a \frac{u}{v}.$$

If $z < 1$, the final member of (1) is less than $1 + (u-v)z/v$, and the character of the inequalities in (2) is reversed; but, as $z^v - 1$ negative, the inequalities

are again reversed on multiplying by z^v-1 , and the result is the same as before.

THEOREM II. *If x lies between 0 and 1, then $(1+a)^x < 1+ax$.*

Let $x = \frac{v}{u}$, $v < u$, and put $(1+a)^{v/u} = 1+y$. Then

$$1+a = (1+y)^{u/v} > 1 + \frac{u}{v}y. \quad (\text{Theorem I}).$$

$$\therefore y < a \frac{v}{u}; \text{ and, therefore, } (1+a)^{v/u} < 1 + a \frac{v}{u}.$$

THEOREM III. *If x is any positive quantity, then $(1+a)^{-x} > 1-ax$.*

First. Let x be greater than 1, and equal to u/v . Put $(1+a)^{-x} = 1+y$.

Then,

$$(1+a)^{-x} = \left(\frac{1}{1+a}\right)^{u/v} = \left(1 - \frac{a}{1+a}\right)^{u/v} > 1 - \frac{a}{1+a} \frac{u}{v} \quad (\text{Theorem I}).$$

$$\therefore 1+y > 1 - \frac{a}{1+a} \frac{u}{v}; \quad y > -\frac{a}{1+a} \frac{u}{v} > -a \frac{u}{v};$$

$$\text{and } (1+a)^{-x} > 1 - a \frac{u}{v}, \text{ or } > 1-ax.$$

(If $a > 0$, then $\frac{a}{1+a} < a$, and $-\frac{a}{1+a} > -a$. If $a < 0$ and equal to $-b$, then,

since $1+a$ is supposed > 0 , $-\frac{a}{1+a} = \frac{b}{1-b} > b$, or $> -a$).

Second. Let x be less than 1, and equal to v/u . Then

$$(1+a) = (1+y)^{-(u/v)} > 1 - y \frac{u}{v} \quad (\text{by preceding case}).$$

Whence, as before,

$$-y \frac{u}{v} < a, \quad y > -a \frac{v}{u}, \text{ or } > -ax.$$

THEOREM IV. *For all values of x greater than 1 or less than 0, $(1+a)^x < 1+ax(1+a)^{x-1}$; and for all values of x between 0 and 1, $(1+a)^x > 1+ax(1+a)^{x-1}$.*

If $x > 1$, or < 0 , we have (Theorems I and III),

$$\left(\frac{1}{1+a}\right)^x = \left(1 - \frac{a}{1+a}\right)^x > 1 - ax \frac{1}{1+a}.$$

$$\therefore 1 > (1+a)^x - ax(1+a)^{x-1}, \quad (1+a)^x < 1+ax(1+a)^{x-1}.$$

The second part of the theorem is similarly proved, by the aid of Theorem II.*

THEOREM V. *For positive values of x , the functions $\left(1 + \frac{1}{x}\right)^x$ and $\left(1 - \frac{1}{x}\right)^x$ are both increasing functions.*

* The demonstration of the theorem just proved the same in principle as that given in Chrystal's *Textbook of Algebra*, and is given here merely for the sake of completeness.

Let x_1 and x_2 be two values of x ; assume $x_1 > x_2$, and put $\left(1 + \frac{1}{x_1}\right)^{x_1} \div \left(1 + \frac{1}{x_2}\right)^{x_2} = \lambda$. Then,

$$\left(1 + \frac{1}{x_1}\right)^{x_1} = \lambda \left(1 + \frac{1}{x_2}\right)^{x_2},$$

whence

$$\lambda^{1/x_2} \left(1 + \frac{1}{x_2}\right) = \left(1 + \frac{1}{x_1}\right)^{x_1/x_2} > 1 + \frac{1}{x_2} \text{ (Theorem I);}$$

and, therefore,

$$\lambda > 1, \text{ or } \left(1 + \frac{1}{x_1}\right)^{x_1} > \left(1 + \frac{1}{x_2}\right)^{x_2}.$$

Similarly for $\left(1 - \frac{1}{x}\right)^x$.

It follows that $\left(1 - \frac{1}{x}\right)^{-x}$ increases as x increases.

THEOREM VI. *The functions $\left(1 + \frac{1}{x}\right)^x$ and $\left(1 - \frac{1}{x}\right)^x$ have each a limit as x increases indefinitely, and these limits are the reciprocal of each other.*

For we have,

$$\left(1 + \frac{1}{x}\right)^x \left(1 - \frac{1}{x}\right)^x = \left(1 - \frac{1}{x^2}\right)^x < 1, \text{ but } > 1 - \frac{1}{x} \text{ (Theorem I).}$$

This shows that neither function can increase indefinitely, since they are both increasing functions (Theorem IV).

From the inequalities

$$1 > \left[1 + \frac{1}{x}\right]^x \left[1 - \frac{1}{x}\right]^x > 1 - \frac{1}{x},$$

follows

$$\lim \left[1 + \frac{1}{x}\right]^x \lim \left[1 - \frac{1}{x}\right]^x = 1.$$